Covariant diagonalization of the perfect fluid stress-energy tensor

Alcides Garat[1]

1. Instituto de Física, Facultad de Ciencias, Iguá 4225, esq. Mataojo, Montevideo, Uruguay.

(Dated: April 14th, 2012)

We introduce new tetrads that manifestly and covariantly diagonalize the stress-energy tensor for a perfect fluid with vorticity at every spacetime point. We discuss the origin of inertia in this special case from the standpoint of our new local tetrads.

I. INTRODUCTION

Relativistic fluid dynamics is a subject of relevance in three main astrophysical problems as we learn in references[1,2]. (a) Jets emerging at relativistic speed from the core of active galactic nuclei, from microquasars or gamma-ray bursts. (b) Compact stars and flows around Black Holes. (c) Cosmology. General relativity is necessary only in (b) and (c). In (a) special relativity is sufficient. It is in this context that we introduce a new technique that might render simplification in both mathematical analysis and conceptual understanding. In previous works of the tetrad series we developed a new method to construct new tetrads when second rank antisymmetric fields are present in curved four-dimensional Lorentzian spacetimes. In manuscript[3] it was the electromagnetic field. In this previous work[3] we found new tetrads that introduced maximal simplification in the expression of the electromagnetic field, manifestly and covariantly diagonalized at every point the stress-energy tensor for a non-null electromagnetic field and maximally simplified the Einstein-Maxwell equations. In our present case we are dealing with a fluid where the stress-energy tensor can be described by the following equation,

$$T_{\mu\nu} = (\rho + p)\, u_\mu\, u_\nu + p\, g_{\mu\nu} \ , \tag{1}$$

where ρ is the energy-density of the fluid, p the isotropic pressure and u^μ its four-velocity field, $g_{\mu\nu}$ is the metric tensor. If in addition this fluid has vorticity $\omega_{\mu\nu}$, then we can proceed to build the new tetrads for this particular case following the method developed in[3]. These new tetrads are going to manifestly and covariantly diagonalize the stress-energy tensor (1) at every spacetime event. We carry out this program in section II.

II. THE FLUID TETRADS

We introduce the fluid extremal field or the velocity curl extremal field through the local duality transformation given by,

$$\xi_{\mu\nu} = \cos\alpha\ \ u_{[\mu;\nu]} - \sin\alpha\ \ * u_{[\mu;\nu]}, \tag{2}$$

where $*u_{[\mu;\nu]} = \frac{1}{2}\, \epsilon_{\mu\nu\sigma\tau}\, g^{\sigma\rho}\, g^{\tau\lambda}\, u_{[\rho;\lambda]}$ is the dual tensor of $u_{[\mu;\nu]}$ and the local complexion α is defined through the condition

$$\xi_{\mu\nu} \; * \xi^{\mu\nu} = 0 \; . \tag{3}$$

The identity,

$$A_{\mu\alpha} \, B^{\nu\alpha} - *B_{\mu\alpha} \; * A^{\nu\alpha} = \frac{1}{2} \, \delta_{\mu}^{\ \ \nu} \, A_{\alpha\beta} \, B^{\alpha\beta} \; . \tag{4}$$

which is valid for every pair of antisymmetric tensors in a four-dimensional Lorentzian spacetime[4], when applied to the case $A_{\mu\alpha} = \xi_{\mu\alpha}$ and $B^{\nu\alpha} = *\xi^{\nu\alpha}$ yields the equivalent condition,

$$\xi_{\mu\rho} \; * \xi^{\mu\lambda} = 0 \; . \tag{5}$$

The complexion, which is a local scalar, can then be expressed as,

$$\tan(2\alpha) = - \left(u_{[\mu;\nu]} \, g^{\sigma\mu} \, g^{\tau\nu} \; * u_{[\sigma;\tau]} \right) \; / \; \left(u_{[\lambda;\rho]} \, g^{\lambda\alpha} \, g^{\rho\beta} \, u_{[\alpha;\beta]} \right) \; . \tag{6}$$

After introducing the new velocity curl extremal field we proceed to write the four orthogonal vectors that are going to become an intermediate step in constructing the tetrad that diagonalizes the stress-energy tensor (1),

$$V_{(1)}^{\alpha} = \xi^{\alpha\lambda} \, \xi_{\rho\lambda} \, X^{\rho} \tag{7}$$

$$V_{(2)}^{\alpha} = \xi^{\alpha\lambda} \, X_{\lambda} \tag{8}$$

$$V_{(3)}^{\alpha} = *\xi^{\alpha\lambda} \, Y_{\lambda} \tag{9}$$

$$V_{(4)}^{\alpha} = *\xi^{\alpha\lambda} \; * \xi_{\rho\lambda} \, Y^{\rho} \; , \tag{10}$$

In order to prove the orthogonality of the tetrad (7-10) it is necessary to use the identity (4) for the case $A_{\mu\alpha} = \xi_{\mu\alpha}$ and $B^{\nu\alpha} = \xi^{\nu\alpha}$, that is,

$$\xi_{\mu\alpha} \, \xi^{\nu\alpha} - *\xi_{\mu\alpha} \; * \xi^{\nu\alpha} = \frac{1}{2} \, \delta_{\mu}^{\ \ \nu} \, Q \; , \tag{11}$$

where $Q = \xi_{\mu\nu} \, \xi^{\mu\nu}$ is assumed not to be zero. We are free to choose the vector fields X^{α} and Y^{α}, as long as the four vector fields (7-10) are not trivial. It is clear that if our choice for these fields is $X^{\alpha} = Y^{\alpha} = u^{\alpha}$, then the following orthogonality relations will hold,

$$g_{\rho\mu}\, u^{\rho}\, V^{\mu}_{(2)} = g_{\rho\mu}\, u^{\rho}\, \xi^{\mu\lambda}\, u_{\lambda} = 0 \tag{12}$$

$$g_{\rho\mu}\, u^{\rho}\, V^{\mu}_{(3)} = g_{\rho\mu}\, u^{\rho}\, \ast\, \xi^{\mu\lambda}\, u_{\lambda} = 0\ , \tag{13}$$

because of the antisymmetry of the velocity curl extremal field. A new vector field can be defined through the expression,

$$V^{\alpha}_{(5)} = V^{\alpha}_{(4)}\, (V^{\rho}_{(1)}\, u_{\rho}) - V^{\alpha}_{(1)}\, (V^{\rho}_{(4)}\, u_{\rho})\ . \tag{14}$$

Through the use of the antisymmetry of $\xi_{\mu\nu}$, the condition (5), the identity (11) and the definition of the vectors (7-10), it is simple to prove the following orthogonalities,

$$u_{\mu}\, V^{\mu}_{(5)} = V^{\mu}_{(2)}\, g_{\mu\nu}\, V^{\nu}_{(5)} = V^{\mu}_{(3)}\, g_{\mu\nu}\, V^{\nu}_{(5)} = 0\ . \tag{15}$$

Given that u^{μ}, $V^{\mu}_{(2)}$, $V^{\mu}_{(3)}$ and $V^{\mu}_{(5)}$ are orthogonal, we can now proceed to see that these tetrad vectors covariantly and manifestly diagonalize the stress-energy tensor (1) at every spacetime point,

$$u^{\alpha}\, T_{\alpha}{}^{\beta} = -(\rho + p)\, u^{\beta} \tag{16}$$

$$V^{\alpha}_{(2)}\, T_{\alpha}{}^{\beta} = p\, V^{\beta}_{(2)} \tag{17}$$

$$V^{\alpha}_{(3)}\, T_{\alpha}{}^{\beta} = p\, V^{\beta}_{(3)} \tag{18}$$

$$V^{\alpha}_{(5)}\, T_{\alpha}{}^{\beta} = p\, V^{\beta}_{(5)}\ . \tag{19}$$

Finally, we normalize this local tetrad,

$$U^{\alpha} = u^{\alpha} \tag{20}$$

$$V^{\alpha} = \xi^{\alpha\lambda}\, u_{\lambda}\, /\, (\, \sqrt{u_{\mu}\, \xi^{\mu\sigma}\, \xi_{\nu\sigma}\, u^{\nu}}\,) \tag{21}$$

$$Z^{\alpha} = \ast\xi^{\alpha\lambda}\, u_{\lambda}\, /\, (\, \sqrt{u_{\mu} \ast \xi^{\mu\sigma} \ast \xi_{\nu\sigma} \ast u^{\nu}}\,) \tag{22}$$

$$W^{\alpha} = \Big(V^{\alpha}_{(4)}\, (V^{\rho}_{(1)}\, u_{\rho}) - V^{\alpha}_{(1)}\, (V^{\rho}_{(4)}\, u_{\rho}) \Big)\, /\, \sqrt{V^{\beta}_{(5)}\, V_{(5)_{\beta}}}\ , \tag{23}$$

where, $V^{\beta}_{(5)}\, V_{(5)_{\beta}} = (V^{\beta}_{(4)}\, V_{(4)_{\beta}})\, (V^{\rho}_{(1)}\, u_{\rho})^2 + (V^{\beta}_{(1)}\, V_{(1)_{\beta}})\, (V^{\rho}_{(4)}\, u_{\rho})^2$.

III. CONCLUSIONS

We believe that this new tetrad might bring about simplification in the analysis of astrophysical relativistic problems where vorticity is present, for instance through the Carter-Lichnerowicz equation[1]. Relativistic fluid dynamics addresses astrophysical phenomena directly related to sources of gravitational waves[7,8] where this new tetrad might find applications. On the other hand, following the ideas in[2,5,6] we can see that "inertia here" is produced by the energy-density, pressure, vorticity and gravity itself "there". We can visualize all this through the fluid differential equations in a curved spacetime, and the tetrads (20-23) themselves. Matter contributes through the energy-density, pressure and vorticity to define the local tetrads "here". The velocity curl is explicitly involved in the tetrad vectors construction through the velocity curl extremal field that we also called fluid extremal field. Energy-density, pressure and vorticity define the gravitational field through the solutions to the differential equations. Gravity also produces "inertia here" through the non-linearities of the differential equations where gravity is a source to gravity itself. The metric tensor is also directly involved in the construction of the tetrad vectors. We quote from[5] "Historically, dynamics was bedevilled from its beginning by the invisibility of space and time. Newton (1686) championed the view that space and time, although invisible, do exist and provide the arena within which motion occurs. Leibnitz (1716) argued that there is no such thing as absolute space but only the relative configurations of simultaneously existing bodies and that time is merely the succession of such instantaneous configurations and not something that flows quite independently of the bodies in the universe and their motion. What Leibnitz was advocating was that dynamics should be based exclusively on observable elements; it should not contain elements that are not in principle observable. This, of course, was Mach's standpoint too (Mach 1872), which Einstein (1916) adopted wholeheartedly when developing General Relativity".

REFERENCES

[1] E. Gourgoulhon, *Proceedings of the School Astrophysical Fluid Dynamics, Cargèse, France* (EDP Sciences, 2006).

[2] I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton University Press, 1995).

[3] A. Garat, J. Math. Phys. **46**, 102502 (2005).

[4] C. Misner and J. A. Wheeler, Annals of Physics **2**, 525 (1957).

[5] J. B. Barbour, *Quantum Concepts in Space and Time: Leibnizian Time, Machian Dynamics and Quantum Gravity* (Clarendon Press, Oxford, 1986).

[6] J. B. Barbour and B. Bertotti, Proc. R. Soc. London A **382**, 295 (1982).

[7] N. Andersson and G. L. Comer, Living Rev. Relativity, *Relativistic Fluid Dynamics: Physics for many different scales* (2007). http://www.livingreviews.org/lrr-2007-1.

[8] J. A. Font, Living Rev. Relativity, *Numerical Hydrodynamics in General Relativity* (2003). http://www.livingreviews.org/lrr-2003-4.